Statistical evaluation of extreme ice loads

Lasse Makkonen, Maria Tikanmäki

VTT Technical Research Centre of Finland Ltd, Finland

lasse.makkonen@vtt.fi; maria.tikanmaki@vtt.fi

***Abstract*— The problem of how to estimate the probability of extreme icing events by historical data is crucial in the optimal design of structures in cold regions. For performing the extreme value analysis, numerous different methods are widely used, and several software packages available. There is no consensus on which method should be preferred. Furthermore, there are different criteria in use for the goodness of a method that estimates the cumulative distribution function and the extremes. By using the probabilistically correct goodness criterion, we present a method that provides estimates for the extremes that are considerably better than obtained by the conventional methods. This new method does not require subjective decisions by the user, and is particularly useful for small data sets, such as icing events observed over a short time-period.**

***Keywords*— Extremes; Extreme value analysis; Return period; Ice load; Icing**

I. Introduction

Estimating the probability of very rare extreme icing events by available observed data is required for safe and economical structural design in cold regions. Such estimation is based on the statistical analysis method called the extreme value analysis (EVA). For performing EVA, numerous different methods exist and are discussed in textbooks and review articles [1]–[13]. There exist several commercial and openly available software packages for the purposes of EVA [14], [15].

Conventionally, [2], [16] the extreme value theory is utilized so that the extremes are assumed to be distributed according to one of the asymptotic extreme value distributions, Weibull, Fréchet or Gumbel, depending on an *a priori* assumption of the parent distribution. Nowadays, these distributions are usually combined into the Generalized Extreme Value distribution (GEV) with a probability density function (PDF) as

$$F(x) = \exp\left[-\left(1+\gamma(x-\mu)/\alpha\right)^{-1/\gamma}\right] \quad (1)$$

where $\alpha > 0$ and $1+\gamma(x-\mu)/\alpha > 0$. GEV is a three parameter distribution, the limiting value of which at $\gamma = 0$ is the Gumbel distribution

$$F(x) = \exp\left(-\exp(-y)\right) \quad (2)$$

where y is the reduced variate

$$y = (x-\mu)/\alpha \quad (3)$$

Here, μ is the location parameter and α is the scale parameter. The GEV distribution has a finite upper tail for the shape parameter $\gamma < 0$ (Weibull), whereas for $\gamma > 0$ (Fréchet) and $\gamma = 0$ (Gumbel) there is no upper bound.

The fit to estimate the parameters μ and α is traditionally performed on a Gumbel probability graph, where the ordinate is the reduced variate from the estimated $F(x)$, plotted as $y = -\ln(-\ln(F(x)))$, and the abscissa is the variate value x. This plot transforms the Gumbel distribution model into a line with a slope of $1/\alpha$ and intercept μ/α. Once the fit is made and the parameters μ and α are determined, one can calculate x that corresponds to any probability $F(x) = P$ and the corresponding return period R. Often, one will need to estimate extreme values of X that are higher than included in the data. Graphically this corresponds to extrapolation along the fit on the graph.

There is no universally accepted method of EVA. In contrast, different authors and software provide different methods and often propose several alternative means for the related procedures. Thus, in practice, performing EVA involves considerable subjectivity. EVA is typically performed either by using the Maximum Likelihood Method (MLE) or the Probability Weighted Moments method (PWM), or by associating each observation to its probability and making a fit.

Many theoretically based extreme value distributions exist for the CDF, different plotting formula are proposed in the literature, and numerous methods are in use for the weighing of data. The fitting can be made with respect to either probability or the variable itself. Bayesian methods are also in use. Consequently, there are around numerous ways to perform EVA, which makes it contaminated with significant methodological uncertainties [13], [17]. This may result in serious underestimation of risks when the observed data sets are small.

We have presented an improved and more objective method of EVA [18], which is outlined and applied to ice load assessment in this paper. In this new method, called VWLS, none of the selections, discussed above, need to be made by the user. Instead, a three parameter Generalized Extreme Value Distribution (GEV) is applied, the plotting is based on the true rank probabilities, and the weighing and fitting are linked to each other by solving their optimal selection iteratively. We show by Monte-Carlo simulations that this method outperforms the other widely used methods of EVA, including the Maximum likelihood (MLE) and probability weighted moments (PWM).

II. The New EVA Method

A. The Fitted Distribution

In the method proposed, GEV is used and all its three parameters are varied. This is not based merely on an assumption of full convergence towards the asymptotic extreme value distribution, but rather that also in the case of a so-called penultimate distribution, extreme value data can be fitted by a three-parameter GEV very well [19].

B. The Plotting Positions

The cumulative distribution function of N number of annual maxima is estimated from the order statistics so that each maximum is assigned a rank m in ascending value $1 \le m \le N$. For a conventional analysis of order ranked extremes, one needs to associate them to some probability, the so-called plotting position. This is done here by the formula

$$P_m = m/(N+1) \tag{4}$$

It was shown [20] that this, so called Weibull plotting position, provides the correct rank probability for any distribution of extremes.

C. Weighing and Fitting

There is significant controversy related to the methods of fitting a distribution to extreme value data [13], [21]. Our fitting method deviates from the conventional methods and is described below.

In the conventional fitting methods, the error, i.e. the difference between the data and the model output, is minimized with respect to the probability P. In contrast, in our method the differences are minimized with respect to the variate X. This is motivated by the fact that, in the case of order ranked data, P is not a variate [22] and that Monte-Carlo simulations support this choice [23].

There are number of ways to fit a distribution to the plotted data. In addition to the simple method of least squares, methods that weigh each extreme based to its statistical confidence have been proposed [4], [21], [24]–[27]. It is noteworthy that all curve-fitting techniques implicitly involve giving weights to the data points.

In our method, the weighted least squares are used in making the fit to GEV. As mentioned above, in our fitting method, the errors are minimized with respect to the random variable X, and not P. For this purpose

$$x_m = F^{-1}\left[m/(N+1)\right] = \mu + \frac{\alpha}{\gamma}\left[-1 + \left(-\ln F(x_m)\right)^{-\gamma}\right] \tag{5}$$

Here, x_m is the value of X given by the estimate of the cumulative distribution function F at the probability corresponding to the observation x_m.

In the method, the weights w_m for the data points are given according to the inverse of the variance of x_m. The variance can be calculated when the cumulative distribution function at $F(x_m)$ is known. This is explained below.

Consider the stochastic variable X_m from the order ranked data.

$$x_1 \le x_2 \le \dots \le x_N \tag{6}$$

There are $m - 1$ independent stochastic variables less than and $N - m$ larger than X_m. Hence, the probability density function $f_m(x_m)$ of X_m can be determined by [4]

$$f_m(x_m) = \frac{N!}{(m-1)!(N-m)!}F(x_m)^{m-1}\left[1-F(x_m)\right]^{N-m} f(x_m) \tag{7}$$

where m is rank number, N is the total number of variables, $F(x_m)$ is the cumulative distribution function and $f(x_m)$ is the density function. For the GEV-distribution, $F(x_m)$ is given by eq. (1) and

$$f(x) = \frac{1}{\alpha}\left[1+\gamma\left(\frac{x-\mu}{\alpha}\right)\right]^{\left(-1/\gamma\right)-1}\exp\left\{-\left[1+\gamma\left(\frac{x-\mu}{\alpha}\right)\right]^{-1/\gamma}\right\} \tag{8}$$

for $x > \mu - \alpha/\gamma$ when $\gamma > 0$ and for $x < \mu + \alpha/(-\gamma)$ when $\gamma < 0$. Outside this range the density is zero. When $\gamma = 0$, the density function is defined as

$$f(x) = \frac{1}{\alpha}\exp\left[-\left(\frac{x-\mu}{\alpha}\right)\right]\exp\left\{-\exp\left[-\left(\frac{x-\mu}{\alpha}\right)\right]\right\} \tag{9}$$

Hence, the mean of the variable x_m is numerically calculated from the distribution function by

$$\bar{x}_m = \int_{-\infty}^{\infty} x_m f_m(x_m)\, dx_m \tag{10}$$

and the variance σ_m^2 by

$$\sigma_m^2 = \int_{-\infty}^{\infty} \left(x_m - \bar{x}_m\right)^2 f_m(x_m)\, dx_m \tag{11}$$

The weights w_m are then determined by equation

$$w_m = \frac{1}{\sigma_m^2} \tag{12}$$

and are normalized so that

$$\sum_{m=k}^{N} w_m = 1 \tag{13}$$

where k is the index of smallest value of x_k that is taken into account in the fitting, as explained below.

Using these weights, the sum

$$\sum_{m=k}^{N} w_m \left(x_m - x_m\right)^2 \tag{14}$$

is minimized.

Thus, in this method, an estimate of the cumulative distribution function $F(x)$ needs to be applied when determining the weights. On the other hand, the weights are required in determining an estimate of $F(x)$. This is solved by considering both the weights and the fitted function simultaneously in an iteration process.

In some cases, the variable X may have an absolute lower limit according to the fitted GEV. This can sometimes be seen also in real data, for example, for an ice load on a structure, for which the limiting lowest value is zero. Since, in EVA, we are interested in the upper tail of the distribution, a criterion on how close to the theoretical lower limit the smallest observed extremes are is applied, and insignificant data are rejected from the analysis accordingly. This criterion and the iteration process used in the analysis are explained in more detail in [18].

III. Performance of the VWLS Method

D. Evaluation of a Fitting Method

We call our improved EVA method VWLS, since it combines minimization of the variance in x (V), Weibull plotting positions (W) and least squares (LS). For demonstrating the power of VWLS, we investigate how well different methods estimate the true CDF of extremes. To compare the methods, a procedure described in [20] and [23] is used, instead of the conventional comparison of the statistics of the distribution parameters or the quantiles. The reasons for this are outlined in [18] and [23], where it was showed that the goodness of fit in the EVA methods need to be compared, not by the statistics of the parameters g, m, a, nor the quantiles, but by the property for which the CDF estimates are *defined*, i.e. the probability. This can be assessed by considering the frequency of samples falling into equally spaced probability intervals as discussed in [20] and [23].

E. Comparison of EVA Methods

Monte Carlo simulations with 100,000 rounds to calculate the bin frequency and *C* were made by fitting the data using the Maximum Likelihood Method ([28], [29]), the Probability Weighted Moments [26], [30], L-moments [6], [31] and LH2 moments [32], as well as our VWLS method. The random data were produced from Gumbel distributions for demonstration for extremes. The parameters represent typical values in icing data. An example of the distribution of samples into the bins in these simulations is shown in Fig. 1.

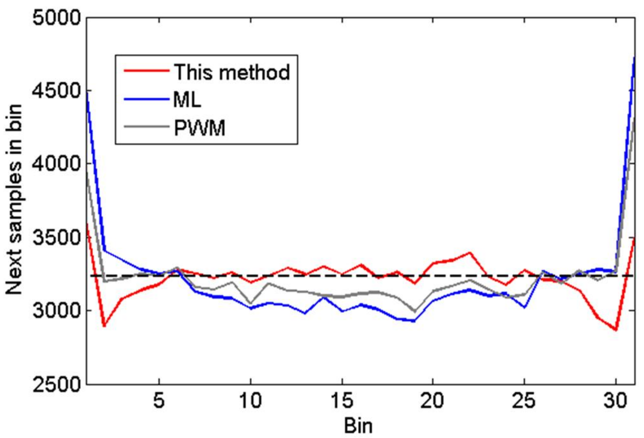

Fig. 1 Distribution of next *m*th samples into the bins that correspond to the *m*th order rank in a Monte-Carlo simulation of 100,000 data sets randomly generated from a Gumbel distribution with a = 10 and m = 70.

The results of the least square error in the bin-frequency for the Gumbel distribution with parameters a = 10 and m = 70 are shown in TABLE I.

The results in TABLE I show that the method VWLS, presented here, outperforms the Maximum Likelihood and Probability Weighted moments clearly up to N = 100. VWLS method also outperforms the L-moments method, for which the results are equal to PWM, and the LH2 moments method. As shown in Fig. 21, for small data sets, VWLS is better particularly in predicting the probability of the largest and smallest value, which are of major interest in e.g. flood frequency analysis. Similar results are obtained for the Frechet and Weibull distributed random samples [18].

TABLE I. Comparisons for the Gumbel Type Distribution with Parameters a = 10 and m = 70. The Columns Show $C\times10^3$ for VWLS, Maximum Likelihood, Probability Weighted Moments, L-moments and LH2 moments methods. The ‘ideal’ Column Presents $C\times10^3$ when Compared with the Actual Distribution.

N	Ideal	VWLS	ML	PWM	L	LH2
10	5	11	65	44	44	64
15	2	12	43	27	27	43
20	3	10	33	21	21	35
30	3	8	21	13	13	25
50	3	6	11	8	8	15
100	3	5	5	5	5	9

F. Estimation by Extrapolation

The typical use of EVA in making ice load estimates is to extrapolate to a probability lower than those covered by the observed data. For testing our new EVA method, it would be preferable to compare the methods in extrapolation against the selected CDFs directly. This cannot be done by the mean of the simulated values of X_m, where *m* refers to a fixed rank, because $(\bar{x}_m, P_m)$ does not fall on a non-linear CDF. Hence, the best method theoretically would be to compare the methods similarly to what was done above, i.e. by fixing a value of *x* and investigating how well $P(x)$ will be predicted in the mean. However, in most international regulating codes and standards EVA is applied in such way that a return value $x(P)$, that corresponds to a given probability *P*, is estimated, not *vice versa*. This value of *x* is the basic value in standards, such as the ISO and Eurocode and is, therefore, of outmost interest in e.g. engineering design.

For these reasons, we compare here the EVA methods in extrapolation by using the variable value $x_m = x(P_m)$. However, as noted above, comparisons of the EVA methods cannot be done by the mean of x_m. This issue does not concern the median. Therefore, the median of *x* is used as the measure of goodness in the following.

Monte-Carlo simulations were made to see how well the three EVA methods estimate the median of the variable value $x_{30/31} = F^{-1}(1 - 1/31)$ from fits to samples of size N = 15. In other words, we extrapolated from each individual fit to the 15 observations, obtained in Section 3.1, and the median of $x_{30/31}$ was calculated. These median values are shown in Table 4 for the five EVA methods. Next, 100,000 sets of 30 samples were generated by a Monte-Carlo simulation from the same selected distributions. For the largest values, at N = 30, the median was also determined, and is marked as ‘From the distribution’ in TABLE II.

TABLE II. Median of the variable value $x_{30/31}$ determined from three extreme value distributions (a = 10; m = 70) and by extrapolating from the fits obtained by the five EVA methods for N = 15.

Distribution type	From the distribution	VWLS	ML	PWM	L	LH2
Fréchet, g = 0.3	**139.9**	139.4	125.7	120.7	120.8	118.9
Weibull, g = -0.3	**92.6**	92.8	89.5	91.5	91.5	91.3
Gumbel	**107.7**	108.1	101.1	102.6	102.7	101.8

TABLE II indicates that the VWLS method proposed here is quite accurate in estimating the return value of an extrapolated extreme in terms of its median. The VWLS method clearly outperforms the Maximum Likelihood method for all three types of extreme value distributions. TABLE II also shows that PWM and L-moments perform similarly, and LH2-moments slightly worse than them. VWLS outperforms PWM, L and LH2 methods as well.

IV. Application: Ice Storm of 1998

We demonstrate the VWLS method by applying it to the data used in assessing the return period of the 1998 Ice Storm in North America. These data were analysed and the problems in the analysis pointed out earlier by [33], who used a fully empirical fit in his extreme value analysis. The result of the analysis by VWLS and three conventional extreme value analysis methods for these data are shown in Fig. 2.

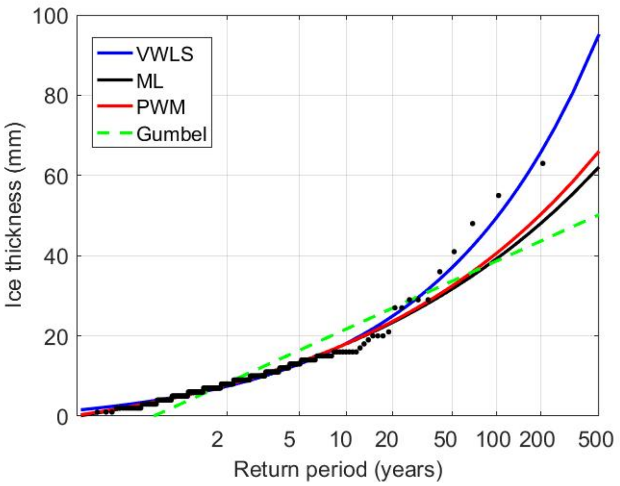

Fig. 2 Comparison of the conventional extreme value analysis methods and the new VWLS method for the radial thickness of ice caused by freezing rain on power line cables in the upper St. Lawrence River Valley in 1958–1998.

The ice thickness in Fig. 2 was simulated by an icing model [34], [35] based on weather data of five statistically independent weather stations each having observations from about 40 years. The stations were combined into a superstation [36], which then included data that corresponds to 206 years.

Fig. 2 shows that for 50 mm ice thickness the Gumbel method predicts a return period of T = 500 years. For 80 mm ice thickness T = 30 000 years based on this Gumbel analysis. The new VWLS method predicts for 50 mm ice thickness a return period of T = 100 years and for 80 mm ice thickness T = 300 years. The predictions by the Maximum-Likelihood method and Probability Weighted Moments are between these predictions and also underestimate the risk of high ice loads. For high design ice loads they predict return periods more than two times longer than those given by VWLS.

V. Conclusions

The new VWLS method provides estimates for the extreme ice loads that are considerably better than obtained by presently available EVA methods, particularly for small data sets. This improved accuracy is very important in practical risk analysis of ice loads. An additional benefit is that using this method requires no subjective methodological decisions.

Acknowledgements

We thank Matti Pajari for inspiration and many fruitful comments. This work was financially supported by Academy of Finland via ERA-Net LAC Energy grant 311987.

References

[1] W.J. Dixon, "Analysis of Extreme Values," *Ann. Math. Stat.* 21, pp. 488–506. 1950.

[2] E.J. Gumbel, *Statistics of Extremes*. New York: Columbia University Press, 1958.

[3] J. Pickands, "Statistical inference using extreme order statistics," *Ann. Stat.* 3, pp. 119–131, 1975.

[4] E. Castillo, *Extreme Value Theory in Engineering*. New York: Academic Press, 1988.

[5] R.L. Smith, "Extreme value analysis of environmental time series: An application in trend detection of ground-level ozone", *Statist. Sci*. 4, pp. 367–393, 1989.

[6] J.R.M. Hosking and J.R. Wallis, *An Approach Based on L-Moments. In: Regional Frequency Analysis*. Cambridge: Univ. Press. 1997.

[7] S.G. Coles, *An Introduction to Statistical Modeling of Extreme Values*. London: Springer-Verlag, 2001.

[8] J. Beirlant, Y. Goegebeur, J. Segers and J. Teugels, *Statistics of Extremes*. Chichester: Wiley, 2004.

[9] E. Castillo, A.S. Hadi, N. Balakrishnan and J.M. Sarabia, *Extreme Value and Related Models with Applications in Engineering and Science.* New Jersey: Wiley, 2005.

[10] I. Jordaan, *Decisions under Uncertainty*. Cambridge: Univ. Press. 2005.

[11] A.J. McNeil, R. Frey and P. Embrechts, *Quantitative Risk Management: Concepts, Techniques, and Tools*. Princeton: Princeton University Press, 2005.

[12] R.-D. Reiss and M. Thomas, *Statistical Analysis of Extreme Values with Applications to Insurance, Finance, Hydrology and Other Fields*. Boston: Birkhäuser, 2007.

[13] L. Makkonen, "Problems in the extreme value analysis," *Struct. Safety* 30, pp. 405–419, 2008a.

[14] A. Stephenson and E. Gilleland, "Software for the Analysis of Extreme Events: The Current State and Future Directions," *Extremes* 8, pp. 87–109, 2005.

[15] E. Gilleland, M. Ribatet and A.G. Stephenson, "A software review for extreme value analysis," *Extremes* 16, pp. 103–119, 2013.

[16] R.A. Fisher and L.H.C. Tippett, "Limiting forms of the frequency distributions of the largest or smallest members of a sample," *Proc. Cambridge Phil. Soc*. 24, 1928, pp. 180–190.

[17] L. Makkonen, "Plotting positions in extreme value analysis," *J. Appl. Meteorol. Climatol*. 45, pp. 334–340, 2006.

[18] L. Makkonen and M. Tikanmäki, "An improved method of extreme value analysis," *J. Hydrol.* X2, 100012, 2019.

[19] E.M. Furrer and R.W. Katz, "Improving the simulation of extreme precipitation events by stochastic weather generators," *Water Resour. Res*. 44, W12439, 2008.

[20] L. Makkonen, M. Pajari and M. Tikanmäki, "Discussion on Plotting positions for fitting distributions and extreme value analysis," *Can. J. Civ. Eng*. 40, pp. 927–929, 2013.

[21] R.W. Katz, M.B. Parlange and P. Naveau, "Statistics of extremes in hydrology," *Adv. Water Resour*. 25, pp. 1287–1304, 2002.

[22] L. Makkonen and M. Pajari, "Defining sample quantiles by the true rank probability," *J. Prob. Stat.* 326579, 6 p, 2014. doi.org/10.1155/2014/326579.

[23] M. Pajari, M. Tikanmäki and L. Makkonen, "Probabilistic comparison of quantile estimators for continuous random variables," *Comm. Stat - Theory Methods* (in press). 2019.

[24] J. Lieblein, "Efficient methods of extreme value methodology," Natl. Bur. Stand. (US) Rep. NBSIR 75-602, 1974.

[25] J.M. Landwehr, N.C. Matalas and J.R. Wallis, "Probability weighted moments compared with some traditional techniques in estimating Gumbel parameters and quantiles," *Water Resour. Res*. 15, pp. 1055–1064, 1979.

[26] J.R. Hosking, J.R. Wallis and E.F. Wood, "Estimation of the generalized extreme value distribution by a method of probability weighted moments," *Technometrics* 27, pp. 251–261, 1985.

[27] Q.J. Wang, "The POT model described by the generalized Pareto distribution with Poisson arrival rate," *J. Hydrol*. 129, pp. 263–280, 1991.

[28] R.A. Fisher, "On the mathematical foundations of theoretical statistics," *Phil. Trans. Roy. Soc*., Ser. A, 222, pp. 309–368, 1922.

[29] J.R. Hosking, "Maximum-likelihood estimation of the parameters of the generalized extreme-value distribution," *Appl. Stat*. 34, pp. 301–310, 1985.

[30] J.A. Greenwood, J. Landwehr, N.C. Matalas and J.R. Wallis, "Probability weighted moments: Definition and relation to parameters of several distribution expressible in inverse form," *Water Resour. Res*., 15, pp. 1049–1054, 1979.

[31] J.R.M. Hosking, "L-moments: Analysis and estimation of distributions using linear combinations of order statistics," *Roy. Stat. Soc. London*, 52, pp. 105–124, 1990.

[32] Q.J. Wang, "LH-moments for statistical analysis of extreme events," *Water Resour. Res*. 33, pp, 2841–2848, 1997.

[33] L. Makkonen, "Extreme value analysis of icing events," in *Proceedings, 10th International Workshop on Atmospheric Icing of Structures (IWAIS)*, 2002, Paper No. 8-6, 6 p.

[34] L. Makkonen, "Modeling power line icing in freezing precipitation," *Atmos. Res*. 46, pp. 131–142, 1998.

[35] K. Jones, "A study of extreme ice loads on power lines in the St. Lawrence River Valley Region," Report, U.S. Army Cold Regions Res. Eng. Lab., Hanover, NH, 2000.

[36] J.A. Peterka, "Improved extreme wind prediction for the United States," *J. Wind Eng. Ind. Aerodyn*. 41-44, pp. 533–541, 1992.