Statistical evaluation of extreme ice loads

Lasse Makkonen, Maria Tikanmäki

VTT Technical Research Centre of Finland Ltd, Finland

lasse.makkonen@vtt.fi; maria.tikanmaki@vtt.fi

Abstract—The problem of how to estimate the probability of extreme icing events by historical data is crucial in the optimal design of structures in cold regions. For performing the extreme value analysis, numerous different methods are widely used, and several software packages are available. There is no consensus on which method should be preferred. Furthermore, there are different criteria in use for the goodness of a method that estimates the cumulative distribution function and the extremes. By using the probabilistically correct goodnes criterion, we present a method that provides estimates for the extremes that are considerably better than obtained by the conventional methods. This new method does not require subjective decisions by the user, and is particularly useful for small data sets, such as icing events observed over a short time-period.

Keywords—Extremes; Extreme value analysis; Return period; Ice load; Icing

I. INTRODUCTION

Estimating the probability of very rare extreme icing events by available observed data is required for safe and economical structural design in cold regions. Such estimation is based on the statistical analysis method called the extreme value analysis (EVA). For performing EVA, numerous different methods exist and are discussed in textbooks and review articles [1]–[13]. There exist several commercial and openly available software packages for the purposes of EVA [14], [15].

Conventionally, [2], [16] the extreme value theory is utilized so that the extremes are assumed to be distributed according to one of the asymptotic extreme value distributions, Weibull, Fréchet or Gumbel, depending on an a priori assumption of the parent distribution. Nowadays, these distributions are usually combined into the Generalized Extreme Value distribution (GEV) with a probability density function (PDF) as

\[
F(x) = \exp \left( \left(1 + \frac{g(x - \mu)}{\alpha} \right)^{-\frac{1}{\gamma}} - 1 \right) \quad (1)
\]

where \( \alpha > 0 \) and \( 1 + g(x - \mu)/\alpha > 0 \). GEV is a three parameter distribution, the limiting value of which at \( \gamma = 0 \) is the Gumbel distribution

\[
F(x) = \exp \left(- \exp \left(- \frac{x - \mu}{\alpha} \right) \right) \quad (2)
\]

where \( y \) is the reduced variate

\[
y = (x - \mu)/\alpha \quad (3)
\]

Here, \( \mu \) is the location parameter and \( \alpha \) is the scale parameter. The GEV distribution has a finite upper tail for the shape parameter \( \gamma < 0 \) (Weibull), whereas for \( \gamma > 0 \) (Fréchet) and \( \gamma = 0 \) (Gumbel) there is no upper bound.

The fit to estimate the parameters \( \mu \) and \( \alpha \) is traditionally performed on a Gumbel probability graph, where the ordinate is the reduced variate from the estimated \( F(x) \), plotted as \( y = - \ln(- \ln(F(x))) \), and the abscissa is the variate value \( x \). This plot transforms the Gumbel distribution model into a line with a slope of \( 1/\alpha \) and intercept \( \mu/\alpha \). Once the fit is made and the parameters \( \mu \) and \( \alpha \) are determined, one can calculate \( x \) that corresponds to any probability \( F(x) = P \) and the corresponding return period \( R \). Often, one will need to estimate extreme values of \( X \) that are higher than included in the data. Graphically this corresponds to extrapolation along the fit on the graph.

There is no universally accepted method of EVA. In contrast, different authors and software provide different methods and often propose several alternative means for the related procedures. Thus, in practice, performing EVA involves considerable subjectivity. EVA is typically performed either by using the Maximum Likelihood Method (MLE) or the Probability Weighted Moments method (PWM), or by associating each observation to its probability and making a fit.

Many theoretically based extreme value distributions exist for the CDF, different plotting formula are proposed in the literature, and numerous methods are in use for the weighing of data. The fitting can be made with respect to either probability or the variable itself. Bayesian methods are also in use. Consequently, there are around numerous ways to perform EVA, which makes it contaminated with significant methodological uncertainties [13], [17]. This may result in serious underestimation of risks when the observed data sets are small.

We have presented an improved and more objective method of EVA [18], which is outlined and applied to ice load assessment in this paper. In this new method, called VWLS, none of the selections, discussed above, need to be made by the user. Instead, a three parameter Generalized Extreme Value Distribution (GEV) is applied, the plotting is based on the true rank probabilities, and the weighing and fitting are linked to each other by solving their optimal selection iteratively. We show by Monte-Carlo simulations that this method outperforms the other widely used methods of EVA, including the Maximum likelihood (MLE) and probability weighted moments (PWM).

II. THE NEW EVA METHOD

A. The Fitted Distribution

In the method proposed, GEV is used and all its three parameters are varied. This is not based merely on an assumption of full convergence towards the asymptotic extreme value distribution, but rather that also in the case of a
so-called penultimate distribution, extreme value data can be fitted by a three-parameter GEV very well [19].

B. The Plotting Positions

The cumulative distribution function of \( N \) number of annual maxima is estimated from the order statistics so that each maximum is assigned a rank \( m \) in ascending value \( 1 \leq m \leq N \). For a conventional analysis of order ranked extremes, one needs to associate them to some probability, the so-called plotting position. This is done here by the formula
\[
P_m = m/(N + 1)
\]
(4)
It was shown [20] that this, so called Weibull plotting position, provides the correct rank probability for any distribution of extremes.

C. Weighing and Fitting

There is significant controversy related to the methods of fitting a distribution to extreme value data [13], [21]. Our fitting method deviates from the conventional methods and is described below.

In the conventional fitting methods, the error, i.e. the difference between the data and the model output, is minimized with respect to the probability \( P \). In contrast, in our method the differences are minimized with respect to the probability \( P \).

There are number of ways to fit a distribution to the plotted data. In addition to the simple method of least squares, methods that weigh each extreme based to its statistical confidence have been proposed [4], [21], [24]–[27]. It is noteworthy that all curve-fitting techniques implicitly involve giving weights to the data points.

In our method, the weighted least squares are used in making the fit to GEV. As mentioned above, in our fitting method, the errors are minimized with respect to the random variable \( X \), and not \( P \). For this purpose
\[
x_m = F^{-1}[m/(N + 1)] = m + \frac{a}{g} \left[ 1 + \ln F(X_m) \right]^{-\gamma}
\]
(5)
Here, \( x_m \) is the value of \( X \) given by the estimate of the cumulative distribution function \( F \) at the probability corresponding to the observation \( x_m \).

In the method, the weights \( w_m \) for the data points are given according to the inverse of the variance of \( x_m \). The variance can be calculated when the cumulative distribution function at \( F(x_m) \) is known. This is explained below.

Consider the stochastic variable \( X_m \) from the order ranked data.
\[
x_1 \leq x_2 \leq \ldots \leq K \leq x_N
\]
(6)
There are \( m - 1 \) independent stochastic variables less than and \( N - m \) larger than \( x_m \). Hence, the probability density function \( f_m(x_m) \) of \( X_m \) can be determined by [4]
\[
f_m(x_m) = \frac{N!}{(m - 1)!(N - m)!} F(x_m)^{m-1} [1 - F(x_m)]^{N-m} f(x_m)
\]
(7)
where \( m \) is rank number, \( N \) is the total number of variables, \( F(x_m) \) is the cumulative distribution function and \( f(x_m) \) is the density function. For the GEV-distribution, \( F(x_m) \) is given by eq. (1) and
\[
f(x) = \frac{1}{a} \left[ 1 + \frac{g x - \mu}{a} \right]^{-1} \exp \left[ \frac{g x - \mu}{a} \right]^{-\gamma}
\]
(8)
for \( x > -\mu/a \) and for \( x < -\mu/a \) when \( \gamma > 0 \) and for \( x > -\mu/a \) when \( \gamma < 0 \). Outside this range the density is zero. When \( \gamma = 0 \), the density function is defined as
\[
f(x) = \frac{1}{a} \left[ 1 + \frac{g x - \mu}{a} \right]^{-1} \exp \left[ \frac{g x - \mu}{a} \right]^{-\gamma} \exp \left[ \frac{g x - \mu}{a} \right]^{1-\gamma}
\]
(9)
Hence, the mean of the variable \( x_m \) is numerically calculated from the distribution function by
\[
x_m = \sum \frac{x_m f_m(x_m) dX_m}{\sum f_m(x_m) dX_m}
\]
(10)
and the variance \( S_m^2 \) by
\[
S_m^2 = \sum \frac{(x_m - x_m)^2 f_m(x_m) dX_m}{\sum f_m(x_m) dX_m}
\]
(11)
The weights \( w_m \) are then determined by equation
\[
w_m = \frac{1}{S_m^2}
\]
(12)
and are normalized so that
\[
\sum w_m = 1
\]
(13)
where \( k \) is the index of smallest value of \( x_k \) that is taken into account in the fitting, as explained below.

Using these weights, the sum
\[
\sum w_m (x_m - x_n)^2
\]
(14)
is minimized.

Thus, in this method, an estimate of the cumulative distribution function \( F(x) \) needs to be applied when determining the weights. On the other hand, the weights are required in determining an estimate of \( F(x) \). This is solved by considering both the weights and the fitted function simultaneously in an iteration process.

In some cases, the variable \( X \) may have an absolute lower limit according to the fitted GEV. This can sometimes be seen also in real data, for example, for an ice load on a structure, for which the limiting lowest value is zero. Since, in EVA, we are interested in the upper tail of the distribution, a criterion on how close to the theoretical lower limit the smallest observed extremes are is applied, and insignificant data are rejected from the analysis accordingly. This criterion and the iteration process used in the analysis are explained in more detail in [18].
The typical use of EVA in making ice load estimates is to extrapolate to a probability lower than those covered by the observed data. For testing our new EVA method, it would be preferable to compare the methods in extrapolation against the selected CDFs directly. This cannot be done by the mean of the simulated values of $X_m$, where $m$ refers to a fixed rank, because $(x_m, m)$ does not fall on a non-linear CDF. Hence, the best method theoretically would be to compare the methods similarly to what was done above, i.e. by fixing a value of $x$ and investigating how well $P(x)$ will be predicted in the mean. However, in most international regulating codes and standards EVA is applied in such way that a return value $x(P)$, that corresponds to a given probability $P$, is estimated, not vice versa. This value of $x$ is the basic value in standards, such as the ISO and Eurocode and is, therefore, of outmost interest in e.g. engineering design.

For these reasons, we compare here the EVA methods in extrapolation by using the variable value $x_m = x(P_m)$. However, as noted above, comparisons of the EVA methods cannot be done by the mean of $x_m$. This issue does not concern the median. Therefore, the median of $x$ is used as the measure of goodness in the following.

Monte-Carlo simulations were made to see how well the three EVA methods estimate the median of the variable value $x_{30/31} = F^{-1}(1 - 1/31)$ from fits to samples of size $N = 15$. In other words, we extrapolated from each individual fit to the 15 observations, obtained in Section 3.1, and the median of $x_{30/31}$ was calculated. These median values are shown in Table 4 for the five EVA methods. Next, 100,000 sets of 30 samples were generated by a Monte-Carlo simulation from the same selected distributions. For the largest values, at $N = 30$, the median was also determined, and is marked as ‘From the distribution’ in Table II.

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The results in Table I show that the method VWLS, presented here, outperforms the Maximum Likelihood and Probability Weighted moments clearly up to $N = 100$. VWLS method also outperforms the L-moments method, for which the results are equal to PWM, and the LH2 moments method. As shown in Fig. 21, for small data sets, VWLS is better particularly in predicting the probability of the largest and smallest value, which are of major interest in e.g. flood frequency analysis. Similar results are obtained for the Frechet and Weibull distributed random samples [18].
TABLE II indicates that the VWLS method proposed here is quite accurate in estimating the return value of an extrapolated extreme in terms of its median. The VWLS method clearly outperforms the Maximum Likelihood method for all three types of extreme value distributions. TABLE II also shows that PWM and L-moments perform similarly, and LH2-moments slightly worse than them. VWLS outperforms PWM, L and LH2 methods as well.

IV. APPLICATION: ICE STORM OF 1998

We demonstrate the VWLS method by applying it to the data used in assessing the return period of the 1998 Ice Storm in North America. These data were analysed and the problems in the analysis pointed out earlier by [33], who used a fully empirical fit in his extreme value analysis. The result of the analysis by VWLS and three conventional extreme value analysis methods for these data are shown in Fig. 2.

Fig. 2 Comparison of the conventional extreme value analysis methods and the new VWLS method for the radial thickness of ice caused by freezing rain on power line cables in the upper St. Lawrence River Valley in 1958–1998.

The ice thickness in Fig. 2 was simulated by an icing model [34], [35] based on weather data of five statistically independent weather stations each having observations about 40 years. The stations were combined into a superstation [36], which then included data that corresponds to 40 years. The stations were combined into a superstation [36], which then included data that corresponds to 40 years.

Fig. 2 shows that for 50 mm ice thickness the Gumbel method predicts a return period of T = 500 years. For 80 mm ice thickness T = 30,000 years based on this Gumbel analysis. The new VWLS method predicts for 50 mm ice thickness a return period of T = 100 years and for 80 mm ice thickness T = 300 years. The predictions by the Maximum-Likelihood method and Probability Weighted Moments are between these predictions and also underestimate the risk of high ice loads. For high design ice loads they predict return periods more than two times longer than those given by VWLS.

V. CONCLUSIONS

The new VWLS method provides estimates for the extreme ice loads that are considerably better than obtained by presently available EVA methods, particularly for small data sets. This improved accuracy is very important in practical risk analysis of ice loads. An additional benefit is that using this method requires no subjective methodological decisions.

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REFERENCES


